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Optimal Relative Error Rational Approximations to e^x

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In his very fine paper [1], G. Németh solves the problem of finding rational functions $r(x)$ of degree (n, n) which (asymptotically) minimize the *relative* error in approximating e^x over $[-1, 1]$. His analysis makes some very clever and intricate use of the hypergeometric as well as other special functions. Our purpose is to rederive his result in a very direct elementary fashion by using our old construction [2] of rational approximates to e^x. At the same time we derive analogous results for degree (m, n) under appropriately weighted sup norms.

THEOREM 1.

$$\inf_r \| 1 - e^{-x} r(x) \| \sim \varepsilon_n = \frac{1}{2^{2n}(2n+1)!\binom{2n}{n}},$$

where $r(x)$ ranges over the rational functions of degree (n, n) and $\|\cdot\|$ is the sup norm over $[-1, 1]$.

Proof. We introduce the function

$$R(x) = \left| \frac{I_+}{I_-} \right|^2, \qquad I_\pm = \int_0^\infty t^n (t \pm w)^n e^{-t}\, dt,$$
$$w = \frac{e^{i\theta}}{2}, \qquad \cos\theta = x. \tag{1}$$

This is clearly a rational function of degree (n, n) since $|p(e^{i\theta})|^2$ is an nth-degree polynomial in x whenever p is an nth-degree polynomial with real coefficients. We will now show that $R(x)$ satisfies

$$1 - e^{-x} R(x) = (-1)^n \varepsilon_n (T_{2n+1}(x) + o(1)) \tag{2}$$

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from which we can conclude that our $R(x)$ does the job and, by the usual alternating signs argument, that no other rational function can do much better, the only conceivable improvement being in the $o(1)$ term.

To prove (2) then, we observe first that

$$\begin{aligned} I_- - e^{-w}I_+ &= \int_0^\infty t^n(t-w)^n\, e^{-t}\, dt - \int_0^\infty t^n(t+w)^n\, e^{-(t+w)}\, dt \\ &= \int_0^\infty t^n(t-w)^n\, e^{-t}\, dt - \int_w^\infty (t-w)^n\, t^n e^{-t}\, dt \\ &= \int_0^w t^n(t-w)^n\, e^{-t}\, dt \\ &= (-1)^n\, w^{2n+1} \int_0^1 u^n(1-u)^n\, e^{-wu}\, du \end{aligned}$$

and, since $\int_0^1 u^n(1-u)^n\, du = n!\, n!/(2n+1)!$, the integrand having a sharp peak at $\frac{1}{2}$, a standard δ-function argument shows

$$\int_0^1 u^n(1-u)^n\, e^{-wt}\, dt = e^{-w/2}\frac{n!\, n!}{(2n+1)!}\,(1+o(1)),$$

so that we conclude

$$I_- - e^{-w}I_+ = (-1)^n\, w^{2n+1}e^{-w/2}\frac{n!\, n!}{(2n+1)!}\,(1+o(1)). \tag{3}$$

As for I_- itself we have

$$\begin{aligned} e^{w/2}I_- &= \int_0^\infty t^n(t-w)^n\, e^{-(t+w/2)}\, dt = \int_{-w/2}^\infty \left(t^2-\frac{w^2}{4}\right)^n e^{-t}\, dt \\ &= \int_{-w/2}^0 \left(t^2-\frac{w^2}{4}\right)^n e^{-t}\, dt \\ &\quad + \int_0^\infty \left[\left(t^2-\frac{w^2}{4}\right)^n - t^{2n}\right] e^{-t}\, dt + \int_0^\infty t^{2n}e^{-t}\, dt \\ &= 0(1) + 0\left(\int_0^\infty n(t+1)^{2n-2}\, e^{-t}\, dt\right) + (2n)! \\ &= 0(1) + 0\left(ne\int_0^\infty t^{2n-2}e^{-t}\, dt\right) + (2n)! \end{aligned}$$

and so we have

$$I_- = e^{-w/2}(2n)!\,(1+o(1)). \tag{4}$$

Dividing (3) by (4) gives

$$1 - e^{-w}\frac{I_+}{I_-} = (-1)^n\, w^{2n+1}\,\frac{n!\,n!}{(2n)!\,(2n+1)!}(1+o(1)) \tag{5}$$

so that, by the algebraic identity $1-|z|^2 = 2\,\mathrm{Re}(1-z) - |1-z|^2$, we obtain

$$\begin{aligned} 1 - e^{-x}R(x) &= 1 - \left| e^{-w}\frac{I_+}{I_-}\right|^2 \\ &= 2(-1)^n\,\mathrm{Re}\, w^{2n+1}\frac{n!\,n!}{(2n)!\,(2n+1)!}(1+o(1)), \end{aligned}$$

the $|1-z|^2$ being easily absorbed into the $o(1)$ term.

Thus we obtain

$$\begin{aligned} 1 - e^{-x}R(x) &= 2(-1)^n\,\frac{\cos(2n+1)\theta + o(1)}{2^{2n+1}}\,\frac{n!\,n!}{(2n)!\,(2n+1)!} \\ &= (-1)^n\,\varepsilon_n(T_{2n+1}(x)+o(1)), \qquad \text{as desired.} \end{aligned} \tag{6}$$

If we use this same approach for the rationals of degree (m,n) by choosing

$$R(x) = \left|\frac{J}{K}\right|^2, \qquad J = \int_0^\infty (t+w)^m\, t^n e^{-t}\,dt,$$

$$K = \int_0^\infty t^m (t-w)^n\, e^{-t}\,dt$$

then this time we obtain

$$\begin{aligned} K - e^{-w}J &= (-1)^n\, w^{m+n+1}\int_0^1 u^m(1-u)^n\, e^{-wu}\,du \\ &\sim (-1)^n\, w^{m+n+1}\frac{m!\,n!}{(m+n+1)!}\, e^{-(m/(m+n))w}, \end{aligned} \tag{7}$$

and

$$K \sim e^{-(n/(m+n))w}(m+n)!. \tag{8}$$

Thereby we obtain

$$1 - e^{-x}R(x) = 2(-1)^n \operatorname{Re} w^{m+n+1} \frac{m!\, n!}{(m+n)!\,(m+n+1)!} \times e^{((n-m)/(n+m))w}(1+o(1))$$

which is to say

$$\frac{e^x - R(x)}{e^{\alpha x}} = (-1)^n \varepsilon_{m,n} \operatorname{Re} e^{i(\theta(m+n+1)+((n-m)/(n+m))\sin\theta)}(1+o(1)). \quad (9)$$

where

$$\alpha = \frac{3n+m}{2n+2m}, \qquad \varepsilon_{m,n} = \frac{1}{2^{m+n}(m+n+1)!\,\binom{m+n}{m}}.$$

Since $\theta(m+n+1) + ((n-m)/(n+m))\sin\theta$ *increases* from 0 to $(m+n+1)\pi$ as θ goes from 0 to π we read off the requisite sign changes in this error term and we deduce our generalization of Theorem 1, namely,

THEOREM 2.

$$\inf_r \left\| \frac{e^x - r(x)}{e^{\alpha x}} \right\| \sim \varepsilon_{m,n},$$

where $r(x)$ varies over rational functions of degree (m, n),

$$\alpha = \frac{3n+m}{2n+2m} \quad \text{and} \quad \varepsilon_{m,n} = \frac{1}{2^{m+n}(m+n+1)!\,\binom{n+m}{m}}.$$

The unnatural look of the quantity α, however, does tend to make the only "nice" case that of $m = n$, i.e., of our first theorem, but this does reprove the Braess–Newman absolute approximation result, namely,

$$a\varepsilon_{m,n} < \inf_r \|e^x - r(x)\| < A\varepsilon_{m,n}, \qquad a, A \text{ absolute positive constants.}$$

REFERENCES

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2. D. J. NEWMAN, Rational approximations to e^x, *J. Approx. Theory* **27** (1979), 234–235.